Complex Numbers

Lecture by Adam Hesterberg (Notes taken by Joy Zheng)

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Note: A large section of the lecture contained graphs that couldn't be reproduced in these notes.

1 If You're Bored

1. Find all solutions to $(a + bi)^{2006} = a - bi$.
2. Decide how i should be written as a matrix.
3. Find $\cos(10\theta)$ in terms of $\cos(\theta)$.
4. Or find $\cos(100\theta)$ without a calculator. (Hint: $(\cos\theta + i\sin\theta)^{100}$)

2 Polynomials

- Linear equation- has 1 root, horizontal lines have 0 roots (excluding $y = 0$).
- Quadratics- has 2 or 0 roots when counting with multiplicity.
  - can find roots using quadratic formula ($ax^2 + bx + c \Rightarrow \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$)
  - $x^2 + 2bx + b^2 = (x + b)^2 \Rightarrow x^2 + 2bx + b^2 - c^2 = (x + b)^2 - c^2, x = \pm c - b$
  - $b^2 - 4ac$ determines the number of roots
  - In order to give all quadratics 2 roots, we declare $i = \sqrt{-1}$. This makes the quadratic equation valid for all quadratics.

3 Complex Number Operations

- Addition/Subtraction: $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$
- Multiplication: $(a + bi) \times (c + di) = ac + bic + adi + bidi = (ac - bd) + (bc + ad)i$
  (Distributivity, Commutativity, Associativity)
  - $(1 + i)^2 = 2i$
  - $(1 + i)^3 = 2i - 2$
  - $(1 + i)^4 = -4$
- Division:
  Division in standard form is messy:
  - $\dfrac{a + bi}{c + di}$

- $\dfrac{(a+bi)(c-di)}{(c+di)(c-di)}$
- $\dfrac{(a+bi)(c-di)}{c^2+d^2}$
- $\dfrac{1}{a+bi} = {\dfrac{a-bi}{|a+bi|}}^2$

Division in polar form is easy: $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

4 Polar Coordinates Review/Crash Course

Cartesian Coordinates- (x,y). Cartesian Coordinates work on a rectangular plane.
Polar Coordinates- (r, θ). Polar Coordinates work on a circular plane.

5 Complex Number Plane

The distance from $a+bi$ to the origin is $\sqrt{a^2+b^2}$. The distance of a product of two complex numbers from the origin is equal to the product of the distances of the original complex numbers from the origin.
$|a+bi| = \sqrt{a^2+b^2}$

$$|a+bi|\,|c+di|$$

$$\sqrt{a^2+b^2}\sqrt{c+di}$$

$$\sqrt{(ac-bd)^2+(ad+bd)^2}$$

All the complex numbers with a magnitude of r form a circle around the origin with radius r. Therefore, we also need an angle to determine the exact point.

$$|x| \bullet |y| = |x \bullet y|$$

$$\arg(x \bullet y) = \arg(x) + \arg(y)$$

The argument of the product of two complex numbers is equal to the sum of the two arguments of the original complex numbers. This is also similar to the property of exponents.
To take advantage of the similarity to polar coordinates, we write complex numbers $a+bi$ in the form $r \bullet e^{i\theta}$, where $r = |a+bi|$ and $\left|e^{i\theta}\right|$.

$$(r, \theta) \Rightarrow (r\cos\theta, r\sin\theta)$$

This preserves the various properties we observed: (see also Division)

$$r_1 e^{i\theta_1} \bullet r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

We can use this to derive DeMoivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = e^{i\theta n} = \cos(n\theta) + i\sin(n\theta)$$

6 Some Solutions: If You're Bored

1. Let $a + bi = re^{i\theta}$.

$$(re^{i\theta})^{2006} = re^{-i\theta}$$
$$r^{2006}e^{2007i\theta} = r$$
$$\cos(2007\theta) + i\sin(2007\theta) = 1$$

The solutions are $\frac{2k\pi}{2007}, including k = 0.$

2. We have a $2x2$ matrix with the values $1, 0, 0, 1$ (left to right, going down). The $2x2$ matrix $0, 1, -1, 0$ squared results in this matrix.

3. *See solution for next question*

4. $\cos(100\theta)$ is the real part of $\cos(100\theta) + i\sin(100\theta)$, or cis$100\theta$. By DeMoivre's theorem, cis(nx) =cis$^n(x)$. This is the real of cis(100θ) =cis(θ^{100}). This can be expanded using binomial coefficients. We can cross out every other term, starting with the second, to get the real part of that value. If we wanted the value of $\sin(100\theta)$, we would simply take the imaginary form of this equation.