SEATTLE INFINITY MATH CIRCLE

Mock MathCounts Test 2011

Written by: Albert Chu

MATHCOUNTS

Solutions

SIMC ANSWER KEYS

SPRINT ROUND

1.	1
2.	2/15
3.	1,001
4.	1/12
5.	75 (mph)
6.	16
7.	27 (coins)
8.	32
9.	1236
10.	2012
11.	501 (zeroes)
12.	10 (days)
13.	360 (ways)
14.	30 (triplets)
15.	0
16.	30
17.	99
18.	61
19.	37
20.	50
21.	282
22.	6
23.	3
24.	4
25.	5
26.	$10\sqrt{13}/13$
27.	192
28.	$1/6$
29.	40 (square units)
30.	30 (ways)

TARGET ROUND

1.	2 (ways)
2.	232
3.	2/21
4.	114/5
5.	7 (triangles)
6.	2
7.	5 (ways)
8.	$\frac{1}{4}\left(5\sqrt{2}+\sqrt{10}\right)$

Spring Round

1. 2011 is prime and 20011 is not a multiple of 2011, so GCD = 1.

2. $0.1\overline{3} = \frac{1}{10} + \frac{3}{90} = \frac{12}{90} = \frac{2}{15}$

3. 1,001 is a palindrome and is also a multiple of 7.

4. $p_3(12) = \frac{1}{36}$, $p_{12}(3) = \frac{2}{36}$, so the sum $p_3(12) + p_{12}(3) = \frac{1}{36} + \frac{2}{36} = \frac{3}{36} = \frac{1}{12}$.

5. $\frac{240}{60} = 4\ hours,\ \frac{160}{80} = 2\ hours, \frac{700-(240+160)}{10-(4+2)} = \frac{300}{4} = 75\ (mph)$

6. $c^2 - a^2 = (c-a)(c+a) = 4(a+4+a) = 4(2a+4) = 128$. Then, $a = 14$, so $b = 16$.

7. $\frac{11.07}{0.25+0.1+0.05+0.01} = \frac{11.07}{0.41} = 27\ (coins)$

8. $a = b = 16$. So, $a + b = 32$.

9. The smallest 4-digit integer is 1,236.

10. The number k has 2010 digit 1's and unit digit 2. The sum is 2012.

11. $\left\lfloor\frac{2011}{5}\right\rfloor = 402$, $\left\lfloor\frac{402}{5}\right\rfloor = 80$, $\left\lfloor\frac{80}{5}\right\rfloor = 16$, $\left\lfloor\frac{16}{5}\right\rfloor = 3$. Then, $402 + 80 + 16 + 3 = 501$.

12. 60 man-days to lay 1000 bricks. It takes 300 main-days to lay 5000 bricks. $\frac{300}{30} = 10\ (days)$.

13. $\frac{1}{2}\cdot\frac{7!}{7} = \frac{1}{2}\cdot 6! = 360\ (ways)$

14. There are cases: 1,1,40 (3 cases); 1,2,20 (6 cases); 1,4,10 (6 cases); 1,5,8 (6 cases); 2,2,10 (3 cases); 2,4,5 (6 cases). So, totally there are 30 cases.

15. $S_x = 0 + 3 + (-3) = 0$ and $S_y = 0 + 3 + 3 = 6$, so $S_x S_y = 0 \cdot 6 = 0$.

16. (13, 5, 12) is a right triangle, so the area is $\frac{5*12}{2} = 30$.

17. $n^4\ (mod\ 10) \equiv 1$ for $n = 1, 3, 7, 9$ and $n^4\ (mod\ 10) \equiv 6$ for $n = 2, 4, 6, 8$.

Therefore, the sum $S = \big((1+1+1+1) + 5 + (6+6+6+6)\big) * 3 = 33 * 3 = 99$

18. $31^{42} = (1+30)^{42} = 1 + 42 * 30 + 30^2(N)$. Therefore, the ten's digit is $2 * 3 = 6$, and unit digit is 1. As a result, the last two digits are 61.

19. The smallest integer is 37.

20. The region R is a square of side length $\frac{10}{\sqrt{2}}$, so the area is $\left(\frac{10}{\sqrt{2}}\right)^2 = 50$.

21. $282 = 93 + 94 + 95$

22. 6 perfect squares: $4^2, 14^2, 24^2, 6^2, 16^2, 26^2$.

23. This function $f(x)$ must have cycles, for instance, $f(0) = 1$ and $f(1) = 0$. $\frac{1}{2}\binom{4}{2} = 3$

24. $a = 1$ and $b = 5$, so $b - a = 4$.

25. A number $N = a_n a_{n-1} \dots a_1 a_0$ is divided by 6. Its remainder is equal to the remainder of $4S + a_0$ divided by 6, where $S = a_n + a_{n-1} + \cdots + a_1$. Since S is a multiple of 3, therefore $4S$ is a multiple of 6. The remainder is therefore equal to the remainder of a_0 divided by 6. The answer is 5.

26. The distance is

$$\frac{20-10}{\sqrt{2^2+3^2}} = \frac{10}{\sqrt{13}} = \frac{10\sqrt{13}}{13}$$

27. $N_0 = 8$, $N_1 = 24$, $N_2 = 24$, $N_3 = 8$. Therefore, $\sqrt{N_0 N_1 N_2 N_3} = 24 * 8 = 192$.

28. The base of the pyramid is $\frac{1}{2}$. The height of this pyramid is 1. Therefore the volume of this pyramid is $\frac{Ah}{3} = \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$.

29. Two vectors are: $(-2, -4)$ and $(-10, 0)$. These parallelograms all have the same area.

$$\begin{vmatrix} -2 & -10 \\ -4 & 0 \end{vmatrix} = 40\ (square\ units)$$

30. $\frac{6!}{24} = 30\ (ways)$

<u>Target Round</u>

1.

$$\frac{4!}{12} = 2\ (ways)$$

2.

Case 1 ($aaaa$): all digits are same, count = 4

Case 2 ($aaab$): 3 digits are same, count = $\binom{4}{1} * \binom{3}{1} * \binom{4}{3} = 48$

Case 3 ($aabc$): 2 digits are same, count = $\binom{4}{1} * \binom{3}{2} * \frac{4!}{2!} = 144$

Case 4 ($aabb$): 2 digits are same and the other 2 digits are also same,

count = $\binom{4}{1} * \binom{3}{1} * \frac{4!}{2!2!} * \frac{1}{2} = 36$

Total count = $4 + 48 + 144 + 36 = 232$

3.

The number of edges is 60. The number of ways choosing two tiles is $\binom{36}{2}$.

$$\text{probability} = \frac{60}{\binom{36}{2}} = \frac{120}{35 * 36} = \frac{2}{21}$$

4.

A trapezoid consists of a parallelogram and a triangle. The triangle has lengths: 3, 4, 5, a right triangle, where 5 = 12 – 7, from the bottom of the trapezoid. The height of this trapezoid is same as the height of the triangle, $\frac{12}{5}$. The area of the trapezoid is

$$\frac{1}{2}(12+7)\frac{12}{5} = 19 * \frac{6}{5} = \frac{114}{5}$$

5.

7 non-congruent triangles, listed below.

2,3,4; 2,4,5; 2,5,6

3,4,5; 3,4,6; 3,5,6

4,5,6

6.

$2^3 = 8 \equiv 1 \ (mod\ 7)$

$2011 = 670 * 3 + 1$

$2^{2011} = 8^{670} * 2 \equiv 1 * 2 \equiv 2 \ (mod\ 7)$

7.

5 ways:

$126 = 41 + 42 + 43$

$126 = 30 + 31 + 32 + 33$

$126 = 15 + 16 + 17 + 18 + 19 + 20 + 21$

$126 = 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18$

$126 = 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16$

8.

$AB^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2$

$AB = 2\sqrt{r_1 r_2}$

Applying this to circles 1 and 3, and to circles 2 and 3, we have

$\sqrt{r_1 r_2} = \sqrt{r_1 r_3} + \sqrt{r_2 r_3}$

$\sqrt{10 r_2} = \sqrt{20} + \sqrt{2 r_2}$

$\sqrt{20} = \sqrt{10 r_2} - \sqrt{2 r_2} = \sqrt{r_2}\left(\sqrt{10} - \sqrt{2}\right)$

$$\sqrt{r_2} = \frac{\sqrt{20}}{\sqrt{10} - \sqrt{2}} = \frac{\sqrt{20}}{8}\left(\sqrt{10} + \sqrt{2}\right) = \frac{1}{4}\left(\sqrt{50} + \sqrt{10}\right) = \frac{1}{4}\left(5\sqrt{2} + \sqrt{10}\right)$$